Spring 2012

Variable screening method using statistical sensitivity analysis in RBDO

Sangjune Bae
University of Iowa

VARIABLE SCREENING METHOD

USING STATISTICAL SENSITIVITY ANALYSIS IN RBDO

by

Sangjune Bae

A thesis submitted in partial fulfillment of the
requirements for the Master of Science degree
in Mechanical Engineering
in the Graduate College of
The University Of Iowa

May 2012

Thesis Supervisor: Professor Kyung K. Choi

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL
_____

MASTER'S THESIS
_____

This is to certify that the Master's thesis of

Sangjune Bae

has been approved by the Examining Committee
for the thesis requirement for the Master of Science
degree in Mechanical Engineering at the May 2012 graduation.

Thesis Committee: _____

Kyung K. Choi, Thesis Supervisor

_____

Jia Lu

_____

Shaoping Xiao

TABLE OF CONTENTS

LIST OF TABLES

v

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In an engineering optimization process, when a problem is formulated, it is necessary to figure out the design variables [1]. Usually, selecting a design variable is somewhat tricky because the problem statement does not provide explicit information regarding design variables. Therefore, one may want to consider every factor in the problem as a design variable so that the result of design optimization could be more trustworthy.

However, considering every single variable will absolutely result in increasing the dimension of a problem. This is critical in the aspect of computational cost in the optimization procedure especially for the reliability-based design optimization (RBDO), where a surrogate model is applied. The RBDO is an optimization algorithm that takes the concept of 'reliability' into account. There are two kinds of the RBDO, namely, sensitivity-based RBDO and sampling-based RBDO [2]. Nowadays, sampling-based RBDO is more widely utilized due to lack of sensitivity information in many applications and the possibility of application of sampling-based RBDO to the broader practical engineering problems. In the sampling-based RBDO, the probability of failure of the constraints needs to be calculated using Monte Carlo Simulation (MCS). Because the sampling method itself contains certain error, a relatively large number of samples are required [3].

Often more than a half million points are required to calculate the probability of failure and sensitivity of the probabilistic constraints. At this stage, the computational cost is highly dependent on the number of design variables because as more design

variables are involved, larger number of samples should be generated to maintain the accuracy of the probability of failure and sensitivity. To get over this curse of dimension, focusing only on important design variables at each design stage is vital. Therefore, sorting out variables of high variance should be carried out by selecting a suitable measure. This process is called 'variable screening'. By setting variables which have large variability as random variables, it is possible to reduce computational cost significantly.

Wei Chen et al. [4] used analysis of variance (ANOVA) decomposition for global sensitivity analysis of the probabilistic characteristics to evaluate relative contribution of not only each design variable, but also interaction of the variables which are called main effects and interaction effects, respectively. Besides, they have made use of variable subset decomposition to converse the calculations of subset interaction sensitivity indices to a combination of subset main sensitivity indices in global sensitivity analysis. K.J. Craig et al. also used a response surface model for variable screening in automotive crashworthiness problem [5]. An explicit surrogate model was also applied in the papers to vivify the importance in the industrial field. One problem in these papers, however, was that only an analytical method was introduced to characterize the importance of the variables, which restricts the benefits of the decomposition method to practical industrial applications.

On the other hand, Y-T Wu et al. [6] have used the local sensitivity of the cumulative density function (CDF) with respect to input parameters as a measure to screen out the variables. Using a sampling method, i.e. the Monte Carlo Simulation, the sensitivity was easily obtained. They have also performed a test-of-hypothesis to

scrutinize the impacts of variables on the CDF sensitivity. The method provided a standard for screening. They have used no approximation either in performance or distribution functions for input variables, which made it applicable to nonlinear functions. But the debate over the number of the MCS samples for the convergence problem still remains controvertible.

In this research, another measurement for screening variables is introduced so that the computational cost is reduced. The statistical inference used to interpret the result of sampling method is introduced, and this notion builds basis for the screening procedure. Moreover, the procedure is designed not to be affected by the form of output functions, which means not only explicit functions, but also implicit constraints are treated. This makes the screening method applicable to the sampling-based RBDO, in which implicit surrogate models could be involved. Besides, the method is applicable to different kind of distributions.

Meanwhile, discussion over the number of MCS samples is presented to try to find the minimum number of samples to obtain meaningful results of the screening procedure. This curiosity naturally draws convergence problem of partial output variance into consideration. The research suggests an iterative method to solve this problem and shows the improvements in the optimization process with suitable examples.

Once the variable screening is complete, the sampling-based RBDO is carried out with the selected design variable set to demonstrate efficiency of the selected variable set in terms of computational cost and satisfaction of constraints.

In Chapter II, a detailed procedure of RBDO is explained with sensitivity derivation. The curse of dimension is introduced which can be involved in RBDO when using a large number of design variables.

In Chapter III, the variable screening method is detailed using the central limit theorem and test-of-hypothesis. The screening method will choose the variable set which affects the output variance the most.

The test-of-hypothesis used in Chapter III is highly dependent on the number of samples. That is, the result of the test can be different if not enough samples are used. Therefore, in Chapter IV, the number of samples is determined which gives a consistent result of the test.

In Chapter V, three examples are demonstrated to verify the necessity of the variable screening method, and RBDO is carried out with the selected variables.

Finally in Chapter VI, the conclusion is provided with advantages and limits of the variable screening method is explained to recommend subsequent future research.

CHAPTER II

SAMPLING-BASED RBDO

In this chapter, the sampling-based RBDO is introduced to motivate exploring the proceeding sections. This chapter is divided into 3 parts. The first part will explain the formulation of the sampling-based RBDO while pointing out the difference between the traditional optimization problem and the RBDO. The second part explains the derivation of the sensitivity of probabilistic constraints using the score functions. On the third part, the problems related to the computational cost are detailed. Mainly, the curse of dimension is explained which calls for a screening method. Different sampling methods will be introduced, and the aftermath caused by the sampling methods is featured.

2.1. Problem Formulation

In general, a traditional deterministic optimization problem is formulated as follows:

$$\begin{aligned}
\text{minimize} \quad & obj(\mathbf{d}) \\
\text{subject to} \quad & H_j(\mathbf{d}) < 0, \qquad j = 1, 2, ..., nc \\
& \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \qquad \mathbf{d} \in R^{ndv}
\end{aligned} \qquad (2.1)$$

where $obj(\mathbf{d})$ is an objective function, $H_j(\mathbf{d})$ are the constraints, $\mathbf{d} = \left[d_i \right]^T$ is a design point, and $\mathbf{d}^L, \mathbf{d}^U$ are the lower and upper bounds of the design point, respectively. The algorithms to find an optimum for this problem are well developed. However, in a practical engineering problem, the design variables involved in the design formulation are mostly not deterministic. That is to say, there exists uncertainty in engineering designs or manufacturing processes [2].

Reflecting these aspects, a RBDO problem considers this uncertainty by introducing the concept of the probability of failure of the constraints at a design point. For a general RBDO problem, the following mathematical formulation can be established:

$$\begin{aligned} \text{minimize} \quad & obj(\mathbf{d}) \\ \text{subject to} \quad & P\left[H_j(\mathbf{x}) > 0\right] \le P_{tar,j}, \; j = 1, 2, ..., nc \\ & \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \quad \mathbf{d} \in R^{ndv} \text{ and } \mathbf{x} \in R^{nrv} \end{aligned} \quad (2.2)$$

where $P[\bullet]$ represents probability, $P_{tar,j}$ is the target probability of the jth constraint, and $\mathbf{x}$ is a random vector. Further, nc stands for the number of constraints, ndv for the number of design variables, and nrv for the number of random variables. The design variable $d_i = \mu(x_i)$ is the mean of a random variable. The number of the random variables is always equal to or larger than the number of design variables. If a random variable is not used as a design variable, the random variable is called 'random parameter'.

In a reliability analysis process, the probability of failure must be calculated. This calculation involves a multi-dimensional integration on the failure domain Φ_F, which is defined as

$$\Phi_F \equiv \left\{ \mathbf{x} \Big| \bigcup_{nc} H_i(\mathbf{x}) > 0 \right\} \quad (2.3)$$

On the domain in Eq. (2.3), it is possible to define an indicator function I_{Φ_F} which possesses the value of one on the failure domain, and zero on the feasible domain as

$$I_{\Phi_F}(\mathbf{x}) \equiv \begin{bmatrix} 1 & \mathbf{x} \in \Phi_F \\ 0 & \text{otherwise} \end{bmatrix} \quad (2.4)$$

This function plays a role of counting the number samples in the failure domain Φ_F. With the indicator function and the domain defined as in Eq. (2.3), and (2.4), respectively, the probability of failure for a design point is computed as

$$
\begin{aligned}
P_{\theta}\left[\mathbf{x} \in \Phi_F\right] &= \int_{\Phi_F} f_{\mathbf{x}}(\mathbf{x};\theta) d\mathbf{x} \\
&= \int_{\mathbb{R}^N} I_{\Phi_F}(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x};\theta) d\mathbf{x} \\
&= E\left[I_{\Phi_F}(\mathbf{x})\right]
\end{aligned}
\tag{2.5}
$$

where f is the probability density function of $\mathbf{x}$, $E[\cdot]$ stands for the mean of arguments, and θ is the parameter vector of the distribution.

Usually, a gradient-based optimization algorithm is used for RBDO. For this algorithm, the function values and their sensitivities at each design point are required. However, in the engineering problems, it is not quite easy to obtain accurate sensitivity due to the absence of explicit functions. In this situation, surrogate models are applied for the optimization process. A surrogate model, also known as a metamodel, is a regression model that estimates an unknown function with a reasonable number of samples. There is a series of surrogate models developed, but for the sampling-based RBDO purpose, Dynamic Kriging (DKG) which is developed by Liang Zhao, et al., is applied [7]. This method approximates the constraints with a dynamic basis selection and is very accurate though it is implicit.

If an accurate surrogate model is available for the optimization purpose, it is possible to obtain the function values and the sensitivities by applying sampling method, i.e. Monte Carlo Simulation (MCS) or the MCS using Latin Hypercube Simulation (LHS) to the surrogate model generated. For the function value evaluation, let $\bar{H}_j(\mathbf{x})$ represent

the surrogate model for the jth constraint. By using a sampling method, the probabilistic constraints in Eq. (2.2) is converted into

$$P\left[H_j(\mathbf{x})>0\right] \cong P\left[\bar{H}_j(\mathbf{x})>0\right] \cong \frac{1}{N}\sum_{i=1}^{N} I_{\bar{\Phi}_F}\left(\mathbf{x}^{(i)}\right) \leq P_{tar,j} \qquad (2.6)$$

where N is the number of samples, $I_{\bar{\Phi}_F}$ is an indicator function defined by the failure domain using the surrogate models as in

$$\bar{\Phi}_F \equiv \left\{\mathbf{x}|\bigcup_{nc} \bar{H}_i(\mathbf{x})>0\right\} \qquad (2.7)$$

and $\mathbf{x}^{(i)}$ is the ith realization of $\mathbf{x}$.

2.2. Sensitivity Analysis

As mentioned above, the sensitivities of probabilistic constraints with respect to the each design variable μ_i are required to use a gradient-based optimization. First, the following regularity conditions must be hold to derive the sensitivity mathematically [8, 9]. For a probabilistic response $H(\boldsymbol{\mu}) == E\left[g(\mathbf{x})\right] = \int_{R^N} g(\mathbf{x})f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\mu})d\mathbf{x}$,

1. The joint PDF of design variables $f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\mu})$ is continuous.

2. $H(\boldsymbol{\mu})$ and $f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\mu})$ are differentiable with respect to their argument $\boldsymbol{\mu}$.

3. The mean satisfies $\boldsymbol{\mu}\in\Omega\subset\mathbb{R}$, where Ω is an open domain.

4. A Lebesgue dominating function $m(\mathbf{x})$ exists and is integrable, which also satisfies the following condition:

$$\left|g(\mathbf{x})\frac{\partial f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\mu})}{\partial\boldsymbol{\mu}}\right| \leq m(\mathbf{x}) \qquad (2.8)$$

Under the regularity conditions, the 1st derivative of the probabilistic constraints

$p(\mathbf{x}) = P_{\boldsymbol{\mu}}\left[y(\mathbf{x}) > 0\right] - P_{Tar}$ with respect to the ith parameter μ_i is obtained as

$$
\begin{aligned}
\frac{\partial P_{\boldsymbol{\mu}}\left[\mathbf{x} \in \Phi_F\right]}{\partial \mu_i} &= \frac{\partial}{\partial \mu_i} \int_{\mathbb{R}^N} I_{\Phi_F}(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\mu}) d\mathbf{x} \\
&= \int_{\mathbb{R}^N} I_{\Phi_F}(\mathbf{x}) \frac{\partial f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\mu})}{\partial \mu_i} d\mathbf{x} \\
&= \int_{\mathbb{R}^N} I_{\Phi_F}(\mathbf{x}) \frac{\partial \log f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\mu})}{\partial \mu_i} f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\mu}) d\mathbf{x} \\
&= E\left[I_{\Phi_F}(\mathbf{x}) \frac{\partial \log f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\mu})}{\partial \mu_i} \right]
\end{aligned}
\tag{2.9}
$$

Here, the distribution parameter vector $\boldsymbol{\theta}$ is $\boldsymbol{\mu}$, which is the design vector. In Eq. (2.9), the 1st order derivative of the log of the PDF is called the first-order score function for μ_i, which is defined as

$$
s_{\mu_i}^{(1)}(\mathbf{x}; \boldsymbol{\mu}) \equiv \frac{\partial \log f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\mu})}{\partial \mu_i}
\tag{2.10}
$$

For the purpose of the sensitivity derivation, as shown above, a differentiation of constraint function is not required. Instead, Eq. (2.9) requires only the 1st derivative of the PDFs. Especially, when a surrogate model is applied to the RBDO problem, the sensitivity of the constraints is usually not available; and also inaccurate even if the surrogate model is considered to be accurate enough. Thus, it is in fact natural to use the score function to obtain the accurate sensitivity.

From the sensitivity which is represented by Eq. (2.9), it is possible to further calculate it by using MCS as

$$\frac{\partial P_{\mathbf{\mu}}\left[\mathbf{x} \in \Phi_F\right]}{\partial \mu_i} = E\left[I_{\Phi_F}\left(\mathbf{x}\right)\frac{\partial \log f_{\mathbf{x}}\left(\mathbf{x};\mathbf{\mu}\right)}{\partial \mu_i}\right]$$
$$\cong \frac{1}{N}\sum_{i=1}^{N}I_{\bar{\Phi}_F}\left(\mathbf{x}^{(i)}\right)s_{\mu_i}^{(1)}\left(\mathbf{x}^{(i)};\mathbf{\mu}\right) \tag{2.11}$$

Deriving the score functions for other distributions is quite straight forward. There is a list of PDFs and their score functions in [10].

2.3. Drawbacks of RBDO - Curse of Dimension

So far, this thesis has presented the formulation of RBDO and the sensitivity derivation. And there the sampling methods i.e. MCS or MCS with LHS were applied for the purpose. The MCS with LHS which uses stratified samples is preferred to the MCS in sampling-based RBDO for its relative accuracy with the small number of samples [11]. However, in the calculation of the probability of failure, it is required to inspect the tail area of a PDF which has very small proportion. Therefore, the number of samples for the calculation is still not small even though the LHS is applied. Further, the LHS is incongruent for the screening purpose in this research because it requires an explicit form of functions when applying the central limit theorem [12].

In the MCS, the average distance between two arbitrary sample points is getting longer if the same number of samples is engaged because the volume of space becomes larger, which in turn requires more samples to maintain the same density. Fig. 2.1 and 2.2 shows a realization of sampling in each dimension. As shown below, they demonstrate the sparseness of samples in the higher dimension. 10 samples were drawn on $\Omega_1 = \left\{x \middle| x \in (0,10)\right\}$ for 1-D case, and the same number of samples were drawn on $\Omega_2 = \left\{(x,y) \middle| x \in (0,10), y \in (0,10)\right\}$.
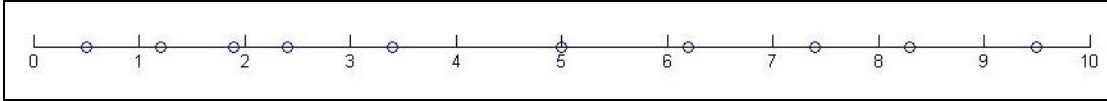
Figure 2.1 Samples in 1-D

The mean distance in 1-D sampling in the case above is $\bar{d}_1 = 3.12$ whereas $\bar{d}_2 = 5.13$ in the 2-D case. Therefore more samples need to be inserted for the 2-D case. For the calculation of the probability of failure in a higher dimension, an extremely large number of samples will be adopted even if the dimension is increased by only 1. As demonstrated, the computational cost for MCS increases exponentially as the dimension increases, and this phenomenon is called 'the curse of dimension' [14].
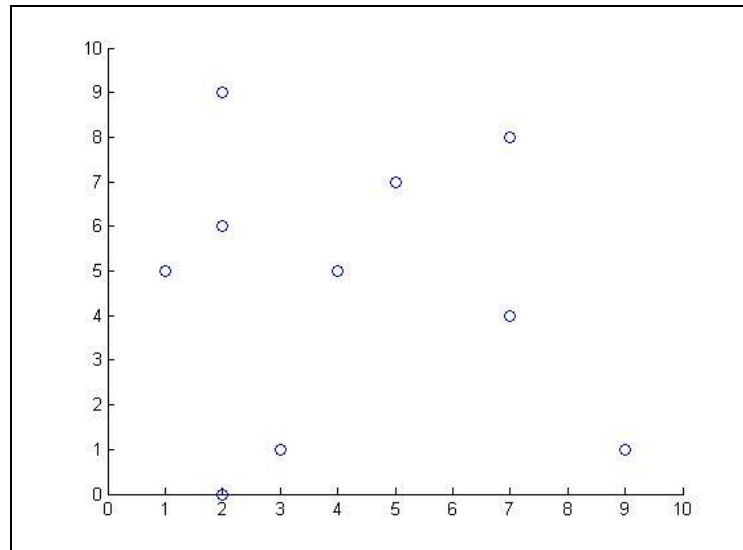
Figure 2.1 Samples in 2-D

Moreover, for a surrogate model generation purpose, the number of function evaluations increases as dimension increases to generate an accurate surrogate model. If a function response is highly nonlinear and the function is of many design variables, then it

easily takes days to generate one surrogate model. In RBDO, when a surrogate model is applied, the models should be generated at each design point. Therefore, the computational time for the whole RBDO procedure is enormous.

To overcome this difficulty, there have been numerous trials such as the dimension reduction method in which the multi-dimensional integrations are decomposed into a number of low dimensional integrations while maintaining a certain level of the accuracy. However, an essential solution for the curse of dimension is to remove the randomness of the unnecessary variables, which is the main purpose of this paper. The method is called 'variable screening,' and it is used to rank the variables in sequence by their relative importance. If the randomness in the specific design variables could be removed, the number of MCS for the RBDO procedure will be reduced significantly. For the surrogate model generation purpose, removing random parameters could reduce the computational cost of the generation. Because the screened variables are assumed to be deterministic, there is no need to carry out the MCS at all.

In the following chapter, different screening methods using statistical inference i.e. the test-of-hypothesis and the central limit theorem are introduced. Especially, the screening methods provided in the subsequent section are able to deal with not only an explicit function but also an implicit function, which is the key feature of this research.

CHAPTER III

VARIABLE SCREENING METHOD

This chapter shows a specific algorithm for the variable screening, and lists the ranking of the design variables determined by their relative importance. The formulation using statistical inferences is introduced with implicit functions, which is a strong aspect of the research. The first part will explain the measures of the screening method and find the possible candidates for screening the variables of the implicit functions. Different measurements with their merits and limits will be included. In the second part, the formulation of the screening method for each measure is explained based on the test-of-hypothesis and the central limit theorem with a detailed procedure.

3.1. Measures for Variable Screening

To perform a variable screening, it is critically important to choose a proper measure. Good measures are those with which it is possible to clearly identify the important variables. A highly strict measure might screen out all variables while a loose measure might not be able to screen out the variables at all. Neither of them is appropriate for the purpose of screening. In fact, the representative values, i.e. the mean, variance, skewness or kurtosis of distributions can be the first guesses for the measures.

Among them, the variance of an output distribution is a suitable measure for variable screening. Let us consider the output distribution of a certain function. If it is possible to quantify the effect of each design variable to the variance of the output distribution, the important variables are those with considerable influence to the output distribution. If variance is chosen as the measure for the variable screening, the selected variable as the result of the variable screening works for the robust design optimization.

Wei Chen et al. [15] used global sensitivity analysis (GSA), which is used to see the influence of variances on the whole output model, by applying analysis of variance (ANOVA) decomposition method to an output function as below. For an explicit function f, the following decomposition is possible:

$$f\left(x_1, x_2, x_3, \ldots x_n\right) = f_m + \sum_{j=1}^{n} \varphi_j\left(x_j\right) + \sum_{k=1}^{n} \sum_{l=1}^{n} \varphi_{kl}\left(x_k, x_l\right) + \ldots + \varphi_{12\ldots n}\left(x_1, x_2, x_3, \ldots x_n\right) \quad (3.1)$$

where f_m is the mean of the output distribution, φ s are decomposition items. In Ref.16, a certain function is decomposed into a number of low dimensional functions, and the global sensitivity is obtained by calculating the partial variances of the decomposed items. The method considers in that not only the effect of a variable itself but also the interaction effect of more than 2 variables could be analyzed. However, their method requires explicit form of constraint function as shown in Eq. (3.1), which is not always available in practical applications since the constraint functions may be implicit.

By analyzing the variance of constraints the variable screening will be carried out in this research. If the variance induced by a variable is small enough to be screened out, then the variable will be regarded as deterministic. Even though the procedure cannot eliminate the randomness in design variables from the initial design, the computational cost will be reduced very much in RBDO process. Even though sensitivity is not calculated, the sensitivity information is embedded in the screening procedure because the output variance depends on the steepness of the constraint functions. The variable screening method to be proposed is applicable where GSA cannot be applied because GSA requires explicit form of constraint functions. The following section shows the formulation of variable screening.

3.2. Formulation for Variable Screening

Consider an implicit constraint function of independent random variables with a certain distribution, denoted as $y = f(\mathbf{x})$, and $\mathbf{x}$ is a random variable vector, defined as

$$\mathbf{x} = [x_1, x_2, ..., x_k]^T \tag{3.2}$$

where k is the number of random variables. Each random variable follows its own distribution, and the parameters of the distribution are considered to be determined in advance. The mean of a distribution is the design variable, and the realization of a design variable is dependent on the type of the distribution.

Out of k number of variables, suppose we want to set the variables which have less impact on the output variance as deterministic. It is possible to figure out the impact of each variable by checking the variance of output response. To screen out the variables, let y_i be the output response induced by the ith variable, while fixing all other variables at their mean as

$$y_i(x_i) = f(\mu_1, \mu_2..., x_i, ...\mu_k) \tag{3.3}$$

to evaluate the effect of the ith variable on the output variance. Because x_i is a random variable, a realization of the output response varies for every repetition. Therefore, it is not enough to evaluate the impact of each variable by evaluating the response only once, which calls for statistical methods to carry out variable screening.

To see the mean of the impact of the ith random variable, a repetition of realizations is needed. For this procedure, MCS is applied. Repeating Eq. (3.3) by m times, it is possible to calculate the sample variance of the output response which is defined as below. The value is referred to as 'the ith partial sample variance' in this

research. One may want to use MCS with LHS instead of the MCS for its efficiency, but MCS with LHS requires an explicit form of f to use the central limit theorem (CLT) [16]. The sample variance is calculated as

$$s_i^2 = \frac{1}{m-1} \sum_{j=1}^{m} \left(f\left(\mu_1, \mu_2 \ldots, x_i, \ldots \mu_k\right) - f(\boldsymbol{\mu}) \right)^2$$
$$= \frac{1}{m-1} \sum_{j=1}^{m} \left(y_i^j - f(\boldsymbol{\mu}) \right)^2 \qquad (i = 1 \sim k) \tag{3.4}$$

where y_i^j is the jth repetition of the response calculation while fixing all the variables other than the ith variable. The sample mean in Eq.(3.4) is divided by $n-1$ by Bessel's correction to make it an unbiased estimator. For all the variables, it is possible to obtain the partial sample variance.

To apply the CLT, the mean of the partial sample variances is required. Therefore, the partial sample variances are calculated p times, resulting in the matrix as

$$\mathbf{M} = \left(s_{ij}^2 \right) = \begin{pmatrix} s_{11}^2 & s_{12}^2 & \cdots & s_{1p}^2 \\ s_{21}^2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ s_{k1}^2 & \cdots & \cdots & s_{kp}^2 \end{pmatrix} \tag{3.5}$$

where s_{ij}^2 is the jth repetition of the ith partial sample variance calculation. Because the elements on the same column are the samples of the partial sample variances, it is possible to apply the CLT by calculating the sample mean of each row. Summing up the elements, one can obtain the sample mean as

$$\bar{s}_j = \frac{1}{p} \sum_{i=1}^{p} s_{ij}^2 \quad \left(j = 1 \sim k \right) \tag{3.6}$$

In fact, in this screening method, relative impacts of the variables are of interest rather than the variance value itself. To evaluate the relative impact, the partial sample variances need to be normalized as

$$\overline{s}_i^N = \frac{\overline{s}_i}{\sum\limits_{j=1}^{k} \overline{s}_j} \qquad (3.7)$$

To apply the test-of-hypothesis, we need to know the distribution that a test statistic follows. However, it is not always possible to figure out which distribution does $\overline{s}_i^N$ in Eq. (3.7) follow because there are k number of variables involved. In fact, it is possible to find out mathematically the form of the distribution by the variable transformation, but the distribution is not a closed form which makes it hard to estimate the percentile values. To overcome this difficulty, the hypothesis would be changed instead of normalizing the random variables.

To determine the distribution of the sample mean, as in Eq. (3.6), CLT is applied. CLT, which is one of the most powerful tools in statistics, states that for any positive integer n, whatever distribution the random variable x follows, for any distribution with finite mean μ and standard deviation, σ the statistic defined as

$$\frac{\sqrt{n}(\overline{x} - \mu)}{\sigma} \sim N(0,1) \qquad (3.8)$$

converges in distribution to a standard normal variable [16].

Usually n in Eq. (3.8), which is the number of repetitions, is required to be larger than 30 to suitably approximate the standard normal distribution. By CLT, $\overline{s}_i$ converges in distribution to $N(\mu, \sigma^2 / n)$. That is,

$$\sqrt{n}\left(\overline{s}_i - \mu_i\right) \xrightarrow{D} N\left(0, \sigma^2\right) \tag{3.9}$$

To identify important design variables, the test-of-hypothesis approach is used. Setting the null hypothesis and the alternative hypothesis as described below, one can conclude that the variable is important if the null hypothesis is rejected within a prescribed confidence. The following hypotheses imply that the null hypothesis is rejected if the normalized partial sample mean is more than x % of the equality of proportion. Because a variable triggering large impact will result in large proportion, the hypothesis is believed to be adequate as

$$H_0 : \mu_i \leq \frac{1}{k} \times \sum_{i=1}^{k} \overline{\mu}_i \times x\%, \qquad H_1 : \mu_i > \frac{1}{k} \times \sum_{i=1}^{k} \overline{\mu}_i \times x\% \tag{3.10}$$

where x is a prescribed percentage value, k is the number of variables, and $\overline{\mu}_i$ are the sample means of the partial sample variances. The region in which the null hypothesis is rejected is called 'critical region'. And the boundary of critical region is referred to as acceptance limit. Because $\overline{s}_i$ follows a normal distribution as in Eq. (3.9), the $1-\alpha$ percent acceptance limit under the null hypothesis can be defined as

$$P\left[\mu_i \leq \frac{1}{k} \times \frac{x}{100} \times \sum_{i=1}^{k} \overline{\mu}_i + Z_{1-\alpha} \cdot \frac{\sigma}{\sqrt{n}}\right] = 1 - \alpha \tag{3.11}$$

where $Z_{1-\alpha}$ is the standard normal variable corresponding to CDF, i.e. $Z_{1-\alpha} = F^{-1}\left(1-\alpha\right)$. Slutsky's theorem [16] makes it possible to substitute the true mean and standard deviation with the sample mean and standard deviation to approximate the distribution. Therefore, it is possible to approximate Eq. (3.11) as

$$P\left[\overline{s}_i \leq \frac{1}{k} \times \frac{x}{100} \times \sum_{i=1}^{k} \overline{\mu}_i + Z_{1-\alpha} \cdot \frac{s}{\sqrt{n}}\right] = 1 - \alpha \tag{3.12}$$

where s is the sample standard deviation. With the substitution, the statistic $\bar{s}_i$ in Eq. (3.12) approximately follows the normal distribution $N\left(\bar{s}_i, s^2\right)$.

If the inequality in Eq. (3.12) is satisfied, then there is no reason to say that the null hypothesis is wrong. In contrast, if the inequality is not satisfied, it is possible to state that the null hypothesis is wrong with the $1-\alpha$ % confidence which means the variable has fairly large impact on the output mean.

When rearranging Eq. (3.12), it is possible to obtain

$$P\left[\bar{s}_i \leq \frac{1}{k} \times \frac{x}{100} \times \sum_{i=1}^{k} \bar{\mu}_i + Z_{1-\alpha} \cdot \frac{s}{\sqrt{n}}\right] = 1-\alpha$$

$$P\left[\left(\bar{s}_i - \frac{1}{k} \times \frac{x}{100} \times \sum_{i=1}^{k} \bar{\mu}_i\right) \middle/ \frac{s}{\sqrt{n}} \leq Z_{1-\alpha}\right] = 1-\alpha \tag{3.13}$$

$q = \left(\bar{s}_i - \frac{1}{k} \times \frac{x}{100} \times \sum_{i=1}^{k} \bar{\mu}_i\right) \middle/ \frac{s}{\sqrt{n}}$ is called 'test statistic', and if q is larger than $Z_{1-\alpha}$ in Eq. (3.13), then the corresponding design variable is selected with the $1-\alpha$ % confidence.

In fact, it is desirable to find the deterministic design optimum (DDO) at first when searching for the RBDO optimum. The research is proposed to be used at the DDO optimum point. Thus, the screening method proposed in this paper is applicable. That is to say, the screening method will filter the important variables, while leaving the other variables at the DDO optimum point. Assuming that inactive constraints at DDO optimum are far enough from the local window, only the active constraints need to be considered for the screening purpose. Therefore, the number of function evaluations in this variable screening method is equal to

$$
\begin{aligned}
(\# \text{ of FE}) = \left(\# \text{ of MCS for } \overline{y}_i\right) \times \left(\# \text{ of repetition for } s^2\right) \\
\times \left(\# \text{ of variables}\right) \times \left(\# \text{ of active constraints}\right) \times \left(\# \text{ of Iterations}\right)
\end{aligned}
\tag{3.14}
$$

As shown in Eq. (3.14), if there are multiple active constraints then the test-of-hypothesis must be done for all the constraints. The variables which will be screened out are those which are screened in every test.

By removing the randomness in the variables that contribute to output variance less, the dimension of the surrogate model is also reduced. The size of the local window to generate a surrogate model is defined by the standard deviation of the input variables. Therefore, if a variable is supposed to be deterministic, the dimension of local window is reduced, which in turn results in reducing the dimension of surrogate model, as well as the computational cost.

The variable screening method does not consider the interaction effect of the design variables, so 1-D surrogate models are used to carry out the variable screening. The computational cost for the generation of 1-D surrogate models is cheaper than the generation of the multi-dimensional surrogate models. Besides, more accurate surrogate model generation is possible because the samples applied to 1-D surrogate model generation are located only on axes whereas samples are located off the axes in multi-dimensional surrogate models. Figure 3.1 and 3.2 demonstrates an example of sample locations in 2-D surrogate model generation and two 1-D surrogate model generations.
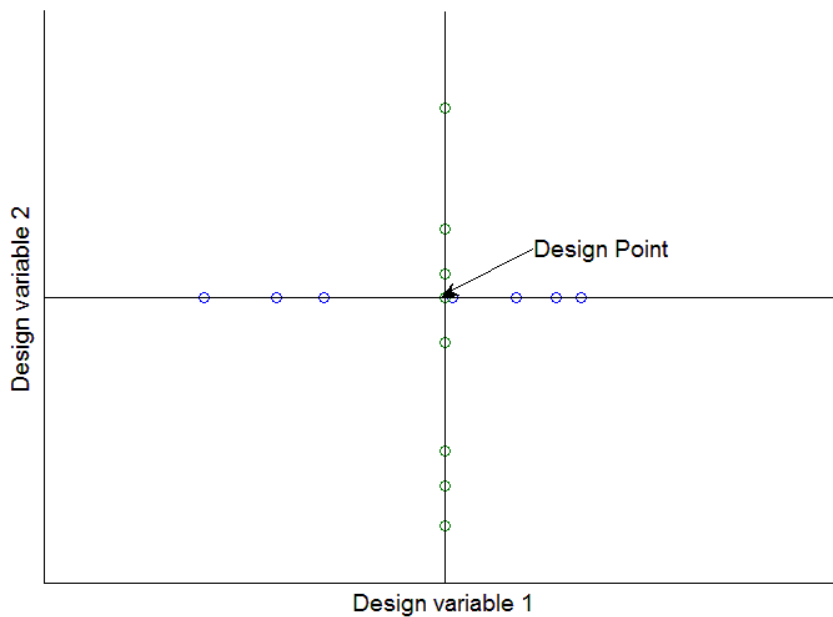
Figure 3.1 Locations of Samples for Surrogate Model Generation for Variable Screening in 2-D
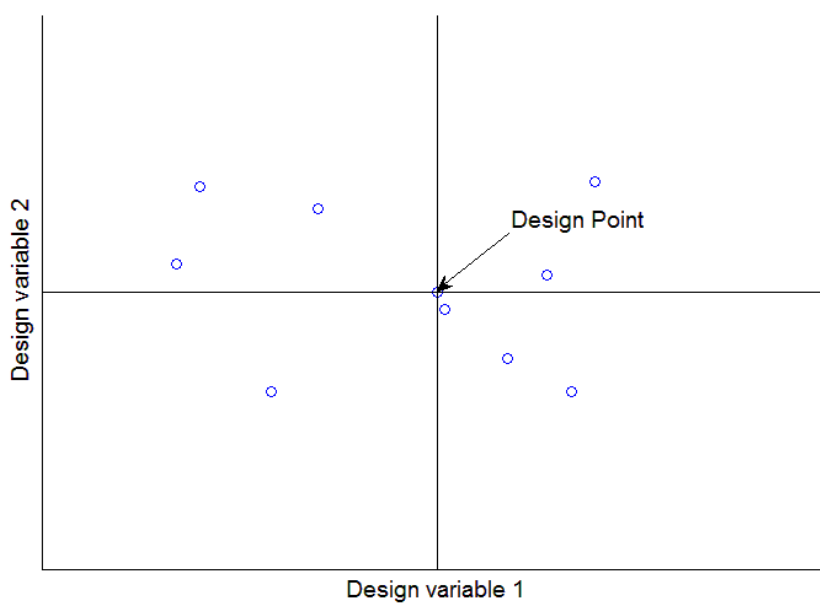
Figure 3.2 Locations of Samples for Surrogate Model Generation for RBDO in 2-D

Besides, if all of the selected design variables are located on the design bound, then the variable screening should be carried out again for the screened variables because the feasible domain may not exist. Likewise, if all of the design variables employed in a constraint are located on the design bound, the variable screening method should not be applied.

CHAPTER IV

SAMPLE SIZE DETERMINATION AND ITS APPLICATION

The errors occur in the test-of-hypothesis is critical in a decision making process. This chapter will describe the possible sources of errors in the test-of-hypothesis and find a way to reduce the possibility of error. In the first part, the problem of convergence related with the number of samples is posted. The second part will be contributed to finding out the minimum number of samples for the required possibility of error. The final part will be devoted to explaining the algorithm of combining the variable screening method and sample size determination.

4.1. Discussion on Convergence

The test-of-hypothesis used in Chapter III is not always correct. There are some components that have effect on decision making. First, acceptance limit in Eq. (3.12) varies with the number of repetition p in the matrix in Eq. (3.5) and the prescribed confidence level. In the Fig. 4.1, the influence of $1-\alpha$ value on the acceptance limit is shown. If a high confidence level is required, then the acceptance region must be large. With a high confidence level, the acceptance limit becomes wider which is consistent with the expectation.
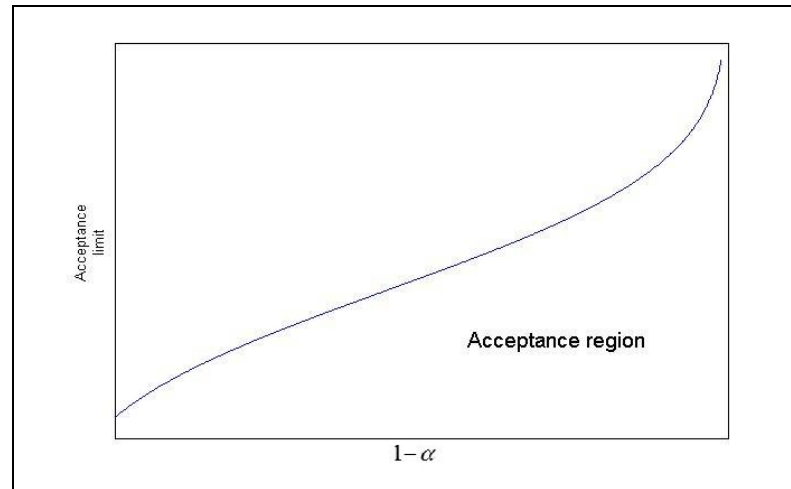
Figure 4.1 Influence of α on Acceptance Limit

If the number of repetition increases, however, the acceptance limit will become narrower because more information is collected. Fig. 4.2 demonstrates the tendency of the acceptance limit variation corresponding to the number of repetitions. As the number of repetitions increases, the acceptance limit tends to converge on a single value.

In fact, errors involved in the hypothesis could be critical in making judgment. There are two kinds of errors, one is choosing the null hypothesis when the alternative hypothesis is true and the other is choosing the alternative hypothesis when the null hypothesis is true. The former error is called Type-I error, and the latter error is called Type-II error. There are 4 states of decision making including errors, as listed in Table 4.1. Type-I error is less significant in the variable screening method because if the error occurs, a variable with less importance would be chosen as an important one. Thus the result of Type-I error is an increment in computational cost. Table 4.2 shows the possibility of each state.
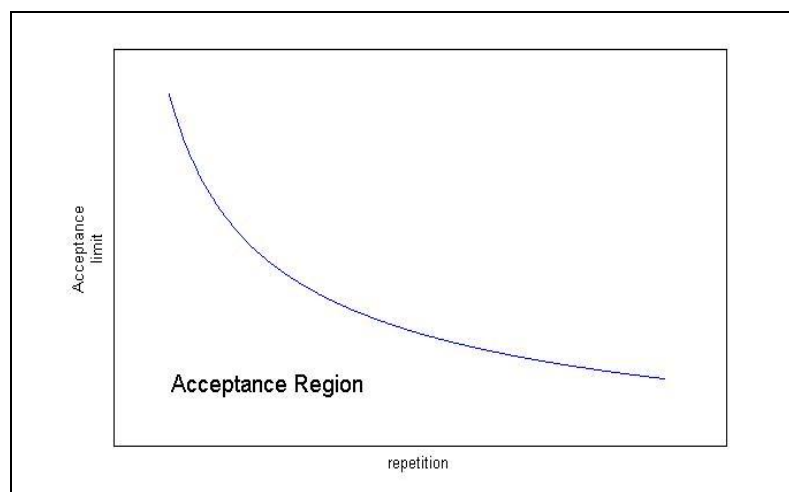
Figure 4.2 Influence of Repetition on Acceptance Limit

However, Type-II error is critical in the optimization process because variables with high impact on the output would be screened out, which may lead to a meaningless optimum result. Therefore, lower probability of Type-II error is preferred. Because there could be different critical regions which are of the same size, one may want to choose the critical region with minimum probability of Type-II error, or equivalently maximizing the power of the test among the critical regions. One of the ways to minimize the probability of Type-II error is to increase the number of samples.

Table 4.1 Decisions in Test-of-Hypothesis

Judgment	State	
	H_0 is true	H_1 is true
Select H_0	Correct	Type-II error
Reject H_0	Type-I error	Correct

The probabilities of the four states in the test-of-hypothesis are shown in Table 4.2. The probability of Type-I error is denoted as α, whereas the probability of Type-II error is denoted as β.

Table 4.2 Probability in Test-of-Hypothesis

Judgment	State	
	H_0 is true	H_1 is true
Select H_0	$1-\alpha$	β
Reject H_0	α	$1-\beta$

The sample mean and sample standard deviation does not converge well with the small number of samples. Therefore, the errors can occur with the small number of samples. Figure 4.3 shows convergence of the sample mean sampled from a normal distribution. As shown in the Fig. 4.3, a sample mean does not converge with the small number of samples but it converges in one value as the number of samples increase. And this fact could induce possible errors on decision making process.
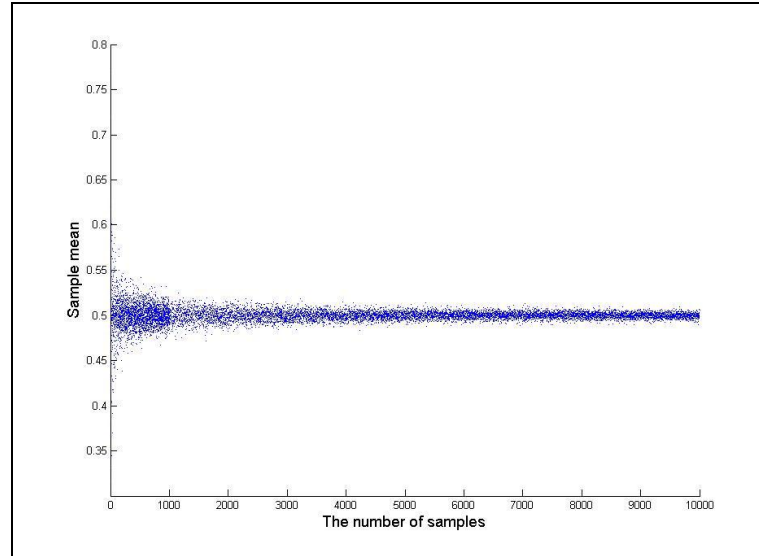
Figure 4.3 Convergence of Sample Mean

However, the variance of the sample mean is high enough to make a decision error in the test-of-hypothesis when a small number of samples are used. When a high confidence level is required for the test, the possibility of making decision error becomes larger. Figure 4.4 and 4.5 show possible Type-I, and Type-II errors on the test-of-hypothesis, respectively. The decision errors are defined as the sample means outside of the acceptance limit when the true mean is inside the limit, or vice versa.
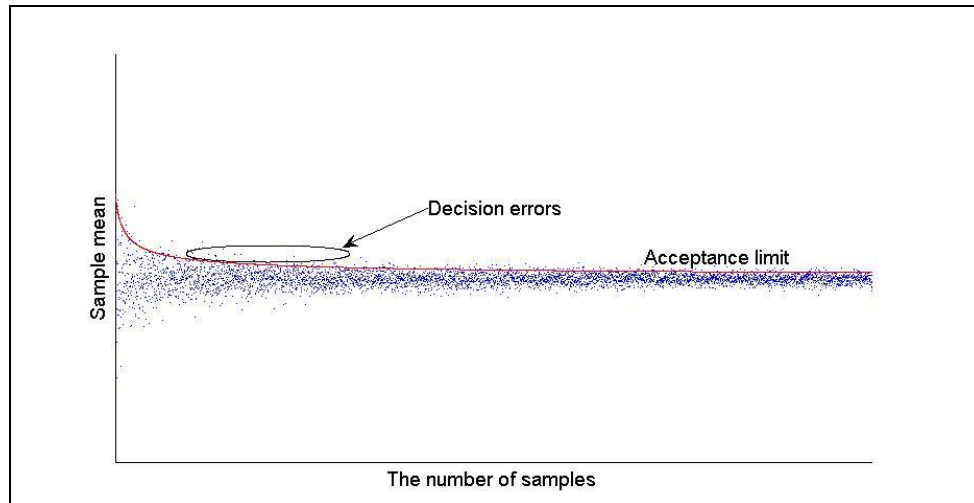
Figure 4.4 Type-I Error
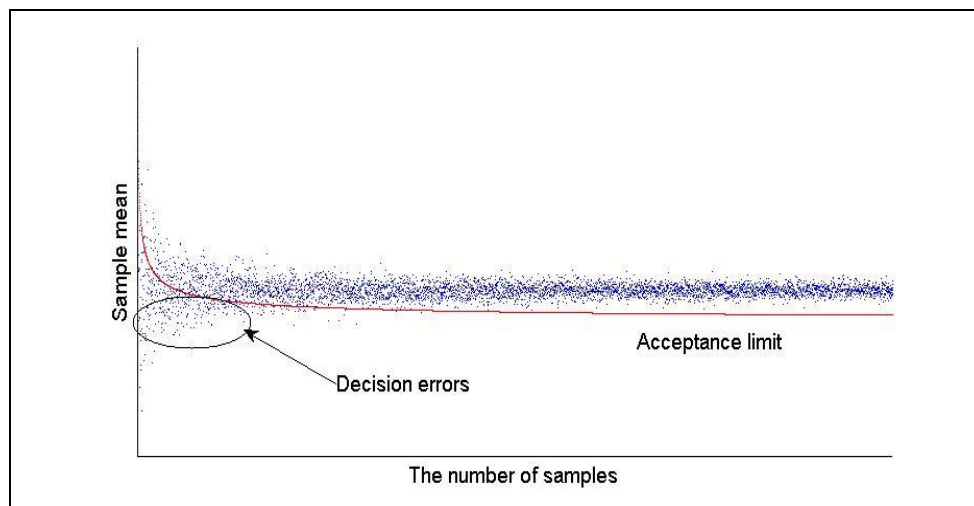
Figure 4.5 Type-II Error

In the test used in this research, there are two possible sources for sampling error. The first is the sample mean defined as in Eq. (3.4), and the other is the sample variance s in Eq. (3.12). As shown on Fig. 4.5 above, it is better to use more samples to reduce the possibility of error. However, a test-of-hypothesis using too many samples will require very large computational cost. Thus, it is necessary to estimate a suitable number for

sampling. Usually, if the null hypothesis is close to the true value or the standard deviation of the test statistic is large, then more samples are required to make the possibility of error small. The following chapter explains how to deal with the convergence problem, and tries to find a suitable number for sampling problem.

4.2. Sample Size Determination

To estimate the number of required samples, we need to investigate the power function, which is defined as

$$
\begin{aligned}
\gamma(\theta) &= 1 - (\text{Type-II error}) \\
&= P_{H_1}\left(\{x_1, x_2, ..., x_n\} \in Cr\right)
\end{aligned}
\tag{4.1}
$$

where θ is a distribution parameter, x_i are random variables, and Cr is critical region. The power function is a function of the distribution parameters, and for the test used in this research, the function is a strictly increasing function with respect to the its parameter μ. Thus, if the difference between the two hypotheses is large, then the power increases as well. This is natural because it is easier to distinguish two hypotheses if the difference is large. Because only the normal distribution is applied to the test-of-hypothesis, it is possible to obtain the power function as

$$
\begin{aligned}
\gamma(\mu) &= P_\mu\left(\frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \geq Z_\alpha\right) \\
&= P_\mu\left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \geq Z_\alpha + \frac{\mu - \mu_0}{\sigma / \sqrt{n}}\right) \\
&= 1 - \Phi\left(Z_\alpha + \frac{\mu - \mu_0}{\sigma / \sqrt{n}}\right) \\
&= \Phi\left(-Z_\alpha - \frac{\mu - \mu_0}{\sigma / \sqrt{n}}\right), \quad \mu \in H_1
\end{aligned}
\tag{4.2}
$$

where Φ is CDF of the standard normal distribution, and Z_α is α-percentile value of the distribution. If the desired power in H_1 is β percent, then the following relation could be obtained:

$$-Z_\alpha - \frac{\mu - \mu_0}{\sigma / \sqrt{n}} = Z_\beta \tag{4.3}$$

Rearranging Eq. (4.3), the number of samples for the desired power is obtained as a function of the alternative hypothesis as

$$n_i = \left[\left(\frac{\sigma_i \left(Z_\alpha + Z_\beta \right)}{\mu_i - \mu_0} \right)^2 \right], \qquad i = 1 \sim r \tag{4.4}$$

where r is the number of test statistics in H_1.

To determine the sample size, μ_i in H_1 should be specified. If the difference in the denominator is small, which means the two hypotheses are ambiguous, then more samples need to be used. Likewise, if the standard deviation is large, the number also needs to be increased. Because H_1 is defined as a range instead of a single value, an estimator for μ and σ should be introduced. As shown in Eq. (4.4), not only μ_i but also σ_i affects the determination. Because of this, in the sample size determination, small $\mu_i - \mu_0$ does not guarantee large n_i. Consequently, the n_i calculated by the test statistics in H_1 need to be compared. Finally, the number of samples will be chosen as the largest value among n_i.

4.3. Algorithm for Variable Screening with Sample Size Determination

So far, a variable screening method using the test-of-hypothesis has been introduced with appropriate sample size determination. However, the result from the test-

of-hypothesis still could be somewhat inconsistent due to the randomness induced by the sample mean and standard deviation in the sample size determination in Eq. (4.4). To overcome this difficulty and obtain a reasonable result, a suitable combination of the two processes is required. In this research, an iterative algorithm is suggested to complete the test which involves a series of tests.
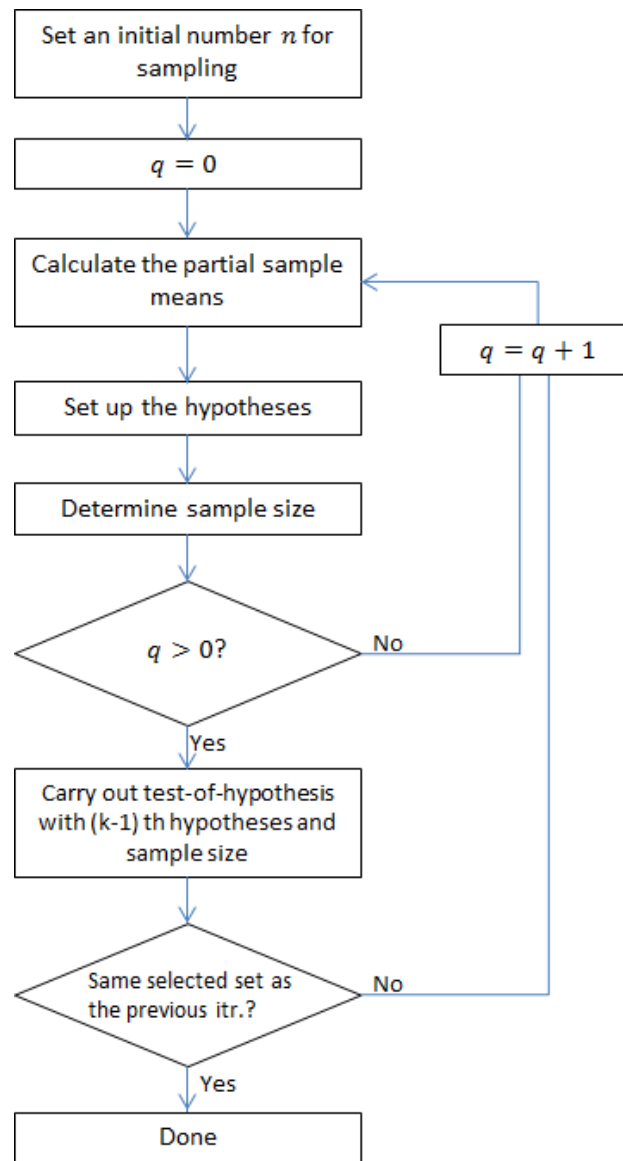
Figure 4.6 Algorithm for Complete Test

Once the partial sample means are calculated, one can state the hypotheses and determine the sample size using the sample means. Therefore, another test-of-hypothesis could be carried out using the hypotheses and the sample size determined from the previous iteration. Thus, iteration consists of a test-of-hypothesis, to set up the hypotheses and determine the sample size. If the result from the current iteration and the previous iteration are the same, then the result from the screening is considered to be accurate enough. Figure 4.6 shows the algorithm for a complete test.

CHAPTER V

EXAMPLES

5.1. Mathematical Verification Example

In this example, the detailed procedure of the variable screening method is shown to verify how the method selects variables which result in large output variance. For the verification purpose, a 10 dimensional function is considered. The function is combination of polynomial functions and an exponential function, which is given as

$$h(\mathbf{x}) = \exp(x_1) + 20x_2^2 + 12x_3 + 5x_4 - 12x_5^4 + x_6^3 + 22x_7^3 + x_8 + 10x_9 + 8x_{10} - 59.71 \quad (5.1)$$

Each random variable follows its own distribution as shown in Table 5.1. Three kinds of probability density functions, namely normal, lognormal, and gamma distributions are considered and their PDF expressions are given in Table 5.2 [16].

Table 5.1 Input Design Variable Information

Variable	Distribution Type	Input Distribution Parameters
x_1	Normal	$\mu = 1,\ \sigma = 0.5$
x_2	Lognormal	$a = -0.018,\ b = 0.1934$
x_3	Normal	$\mu = 1,\ \sigma = 1.1$
x_4	Lognormal	$a = -0.018,\ b = 0.1934$
x_5	Gamma	$\alpha = 25,\ \beta = 0.04$
x_6	Gamma	$\beta = 1.5625,\ \alpha = 0.64$
x_7	Normal	$\mu = 1,\ \sigma = 0.2$
x_8	Gamma	$\alpha = 2.77,\ \beta = 0.36$
x_9	Normal	$\mu = 1\ ,\sigma = 0.85$
x_{10}	Lognormal	$a = -0.0865,\ b = 0.4161$

Table 5.2 PDFs of Normal, Lognormal and Gamma Distribution

Distribution Type	PDF
Normal	$f\left(x;\mu,\sigma\right)=\dfrac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right\},\ -\infty<x<\infty$
Lognormal	$f\left(x;a,b\right)=\begin{cases}\dfrac{1}{xb\sqrt{2\pi}}\exp\left\{-\dfrac{1}{2}\left(\dfrac{\ln x-a}{b}\right)^2\right\} & 0<x<\infty\\0 & \text{elsewhere}\end{cases}$
Gamma	$f\left(x;\alpha,\beta\right)=\begin{cases}\dfrac{1}{\Gamma\left(\alpha\right)\beta^\alpha}x^{\alpha-1}e^{-x/\beta} & 0<x<\infty\\0 & \text{elsewhere}\end{cases}$ $\Gamma\left(\alpha\right)=\int_0^\infty y^{\alpha-1}e^{-y}dy$

The mean and standard deviation for each distribution corresponding to the input parameters in Table 5.1 can be obtained using equations in Table 5.3.

Table 5.3 Mean and Standard Deviation of Normal, Lognormal and Gamma Distribution

Distribution Type	Mean	Standard Deviation
Normal	μ	σ
Lognormal	$\exp\left(a+b^2/2\right)$	$\sqrt{\left(\exp\left(b^2\right)-1\right)\exp\left(2a+b^2\right)}$
Gamma	$\alpha\beta$	$\sqrt{\alpha\beta^2}$

Table 5.4 shows the calculated means and the standard deviations of input variables in Table 5.1. x_3 has the largest input standard deviation whereas x_5 has the smallest one.

Table 5.4 Mean and Standard Deviation of Input Design Variables

Variable	Distribution Type	Mean	Standard Deviation
x_1	Normal	1	0.500
x_2	Lognormal	1	0.195
x_3	Normal	1	1.100
x_4	Lognormal	1	0.195
x_5	Gamma	1	0.2
x_6	Gamma	1	0.8
x_7	Normal	1	0.200
x_8	Gamma	0.9972	0.598
x_9	Normal	1	0.850
x_{10}	Lognormal	1	0.434

In the sample size determination and setting the hypotheses in the iteration 0, 100 samples were used to evaluate the sample mean, and 50 samples were used to calculate the sample standard deviation. The mean values for the partial sample variance are calculated as shown in Table 5.5. At the iteration 0, the variable 7 shows the largest impact on the output variance while the variable 8 shows the smallest impact.

Table 5.5 Mean of Partial Sample Variance in Iteration 0

Variable	x_1	x_2	x_3	x_4	x_5
Mean	2.617	68.756	172.763	0.969	169.128
Variable	x_6	x_7	x_8	x_9	x_{10}
Mean	163.108	198.275	0.352	72.431	12.335

From Table 5.5, for this example, one can set up the null and the alternative hypotheses as

$$H_0^1 : \overline{x} \le 91.7954 \times 100\%, \qquad H_1^1 : \overline{x} > 91.7954 \times 100\% \qquad (5.2)$$

The superscripts in null and alternative hypotheses represent the iteration number in which the hypotheses are used. Eq. (5.2) is calculated from Eq. (3.10), by substituting k with 10, and $\overline{\mu}_i$ by the means in Table 5.5. And Eq. (5.2) becomes the standard for the variable screening process in the next iteration. While doing so, let the probability of Type-I error be 95%. Then Z_α in Eq. (4.4) is equal to $\Phi^{-1}(0.95) = 1.645$, where Φ is the CDF of the standard normal distribution [16].

For the sample size determination, the variables of which the mean is larger than 91.7954 are considered. In this example, the partial sample variances corresponding to the design variables x_3, x_5, x_6 and x_7 are larger than 91.7954. Calculating the standard deviation of the partial sample variance for each variable, it is possible to obtain Table 5.6.

Table 5.6 Standard Deviation of Partial Sample Variance in Iteration 0

Variable	x_1	x_2	x_3	x_4	x_5
Standard Deviation	0.787	15.858	23.140	0.145	75.265
Variable	x_6	x_7	x_8	x_9	x_{10}
Standard Deviation	507.226	44.339	0.074	10.639	2.592

While setting the desired power for the test statistics, which is Z_β in Eq. (4.4), as 50%, it is possible to obtain Table 5.7 by substituting σ_i with the variance of the partial sample variance in Table 5.6, and μ_i with the mean of the partial sample variance in Table 5.5, μ_0 with 91.7954 in Eq. (5.2).

Table 5.7 Sample Size Determination in Iteration 0

Variable	x_3	x_5	x_6	x_7
Sample size	2	13	651	4
Maximum Sample Size	651			

As shown in Table 5.7, the minimum sample size to satisfy the probability of Type-II error in this example is $n = 651$. In the next iteration, the means of the partial sample variance can be calculated again using the sample size $n = 651$ as in Table 5.8.

Table 5.8 Mean of Partial Sample Variance in Iteration 1

Variable	x_1	x_2	x_3	x_4	x_5
Mean	2.683	68.962	175.037	0.965	175.397
Variable	x_6	x_7	x_8	x_9	x_{10}
Mean	157.691	200.290	0.357	71.595	12.140

From Table 5.7, one can set up the null and the alternative hypotheses as

$$H_0^1 : \bar{x} \le 86.6716 \times 100\%, \qquad H_1^1 : \bar{x} > 86.6716 \times 100\% \qquad (5.3)$$

The null and alternative hypotheses in Eq. (5.3) are calculated from Eq. (3.10), by substituting k with 10, and $\bar{\mu}_i$ with the means in Table 5.7. The standard deviations of the partial sample variances are calculated again using the sample size $n = 651$ as in Table 5.9.

Table 5.9 Standard Deviation of Partial Sample Variance in Iteration 1

Variable	x_1	x_2	x_3	x_4	x_5
Standard Deviation	0.744	14.344	24.112	0.154	79.242
Variable	x_6	x_7	x_8	x_9	x_{10}
Standard Deviation	207.387	42.271	0.070	10.092	2.845

The sample size to calculate the partial sample variance in the next iteration is obtained as in Table 5.10.

Table 5.10 Sample Size Determination in Iteration 1

Variable	x_3	x_5	x_6	x_7
Sample size	2	14	461	3
Maximum Sample Size	461			

The 95% percentile value of the standard normal distribution is 1.645 [16]. Therefore, if the test statistic (q), which is calculated from Eq. (3.13) by substituting k with 10, $\frac{1}{k}\sum_{i=1}^{k}\bar{\mu}_i$ with 91.7954 from Eq. (5.2), $\bar{s}_i$ with the means of the partial sample variance from Table 5.7, and s with the standard deviation of the partial sample variance in Table 5.9 is larger than 1.645, the null hypothesis will be rejected and the corresponding design variable will be selected. Table 5.11 shows the calculated value of the test statistics. According to the result, the design variables x_3, x_5, x_6, and x_7 are believed to have large impacts on the output variance.

Table 5.11 Result of the Test-of-Hypothesis in Iteration 1

Variable	x_1	x_2	x_3	x_4	x_5
Test Statistic	-1.9e+03	-14.532	58.650	-8.2e+03	15.147
Variable	x_6	x_7	x_8	x_9	x_{10}
Test Statistic	4.011	45.706	-1.8e+04	-24.388	-435.888

Consequently, only one variable could be considered to be random while the rest are considered as deterministic variables by following this result. To check the

consistency of the result, more iteration is required. Using the maximum sample size in Table 5.10, it is possible to carry out the next iteration. Table 5.12 shows the mean of the partial sample variance in the iteration 2, and Table 5.13 shows the standard deviation of the partial sample variance in the iteration 2.

Table 5.12 Mean of Partial Sample Variance in Iteration 2

Variable	x_1	x_2	x_3	x_4	x_5
Mean	2.625	70.755	173.620	0.944	165.773
Variable	x_6	x_7	x_8	x_9	x_{10}
Mean	166.453	200.651	0.368	71.499	12.609

Table 5.13 Standard Deviation of Partial Sample Variance in Iteration 2

Variable	x_1	x_2	x_3	x_4	x_5
Standard Deviation	0.749	15.256	25.490	0.150	68.626
Variable	x_6	x_7	x_8	x_9	x_{10}
Standard Deviation	222.099	42.956	0.070	10.290	2.759

Table 5.14 shows the result of the second iteration in the variable screening. The result is consistent with the result of the iteration 1, so the iteration stops. According to the result, and x_3, x_5, x_6, and x_7 are required to be random while the others are fixed at the DDO optimum, and thus the result gives a guideline which variables should be removed during the RBDO process.

Table 5.14 Result of the Test-of-Hypothesis in Iteration 2

Variable	x_1	x_2	x_3	x_4	x_5
Test Statistic	-2.6e+03	-23.457	77.384	-1.2e+04	28.009
Variable	x_6	x_7	x_8	x_9	x_{10}
Test Statistic	4.891	61.659	-2.7e+04	-32.564	-575.531

To check whether the variable screening method truly selects the design variables which give large impact on the output variance, the output variance of each term in Eq. (5.1) are calculated by MCS using 1,000,000 samples.

Table 5.15 Output Variance of Each Term

Term	$\mathrm{Exp}(x_1)$	$20x_2^2$	$12x_3$	$5x_4$	$-12x_5^4$
Output Variance	2.704	69.947	174.506	0.955	173.267
Variable	x_6^3	$22x_7^3$	x_8	$10x_9$	$8x_{10}$
Output Variance	154.644	203.029	0.3598	72.360	12.133

According to Table 5.15, x_7 has the largest impact on the output variance whereas x_8 has the smallest impact. The output variances x_3, x_5, x_6 and x_7 are the large, and the variables are selected. Thus the variable screening result is consistent with the result of the Table 5.14. In reality, however, calculating output variance of each term is not possible. Therefore, using the variable screening method can be a solution to select important design variables.

As shown in this example, the variable screening method selects the design variables which have large impact on the output variance.

5.2. Speed Reducer Problem

A speed reducer is to be optimized in this problem [17]. The formulation for this problem is to

Minimize $C(\mathbf{d})$

S.T. $\quad P\big(G_i\big(\mathbf{X}(\mathbf{d})>0\big)\big) \le P_{F_i}^{Tar}, \qquad i = 1 \sim 11$

$\quad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U$

where

$$C(\mathbf{d}) = 0.7854 d_1 d_2^2 \big(3.3333 d_3^2 + 14.9334 d_3 - 43.0934\big)$$
$$- 1.508 d_1 \big(d_6^2 + d_7^2\big) + 7.477 \big(d_6^3 + d_7^3\big) + 0.7854 \big(d_4 d_6^2 + d_5 d_7^2\big)$$

$$G_1(\mathbf{X}) = \frac{27}{X_1 X_2^2 X_3} - 1, \; G_2(\mathbf{X}) = \frac{397.5}{X_1 X_2^2 X_3^2} - 1, \; G_3(\mathbf{X}) = \frac{1.93 X_4^3}{X_2 X_3 X_6^4} - 1$$

$$G_4(\mathbf{X}) = \frac{1.93 X_5^3}{X_2 X_3 X_7^4} - 1, \; G_5(\mathbf{X}) = \frac{\sqrt{\big(745 X_4 / (X_2 X_3)\big)^2 + 16.9 \times 10^6}}{0.1 X_6^3} - 1100$$

$$G_6(\mathbf{X}) = \frac{\sqrt{\big(745 X_5 / (X_2 X_3)\big)^2 + 157.5 \times 10^6}}{0.1 X_7^3} - 850, \; G_7(\mathbf{X}) = X_2 X_3 - 40 \qquad (5.4)$$

$$G_8(\mathbf{X}) = 5 - \frac{X_1}{X_2}, \; G_9(\mathbf{X}) = \frac{X_1}{X_2} - 12, \; G_{10}(\mathbf{X}) = \frac{1.5 X_6 + 1.9}{X_4} - 1$$

$$G_{11}(\mathbf{X}) = \frac{1.1 X_7 + 1.9}{X_5} - 1, \; P_{F_i}^{Tar} = 3\sigma \; \text{design} = 0.00135, \; i = 1 \sim 11$$

There are eleven constraints and seven design variables. The target probability of failure is 0.0135%, which is a 3σ-design. The speed reducer consists of two shafts with bearings and gears. In this optimization problem, the weight of the reducer is to be minimized subject to the contact stress, bending stress, longitudinal displacement of the shafts, stress imposed on the shafts, and geometric constraints.

In Fig 5.1, the location of each design variable is shown [17]. x_1 is the width of the gear and x_2 is the teeth module, which is a scaling factor to enlarge the gear teeth. x_3 is the number of teeth in the pinion, x_3 and x_4 are the distance between bearings, and x_6 and x_7 are the diameters of axis.
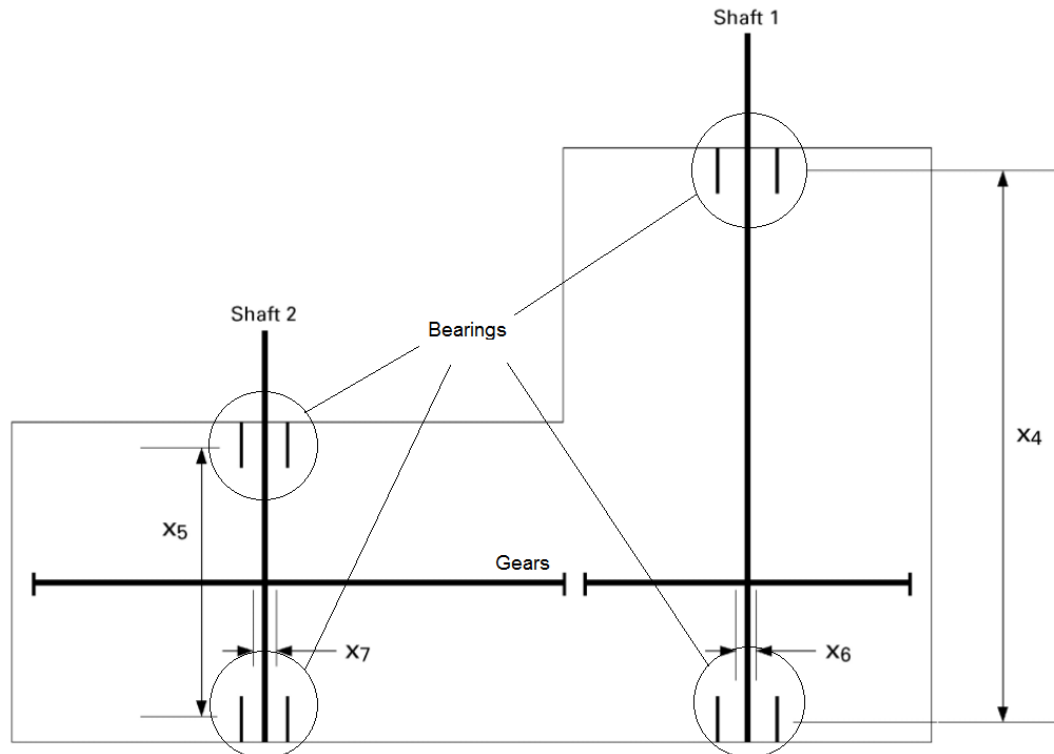
Figure 5.1 Speed Reducer

The distribution type, mean, standard deviation, and design bound of the design variables is shown in Table 5.16. Each design variable follows a normal distribution with different mean and standard deviation.

Table 5.16 Input Design Variable Information

Variable	Distribution Type	Initial Design Parameters	Design Bounds
x_1	Normal	$\mu = 3.2,\ \sigma = 0.005$	$2.6 \le x_1 \le 3.6$
x_2	Normal	$\mu = 0.75\ \sigma = 0.005$	$0.7 \le x_2 \le 0.8$
x_3	Normal	$\mu = 22.5,\ \sigma = 0.005$	$17.0 \le x_3 \le 28.0$
x_4	Normal	$\mu = 7.8,\ \sigma = 0.005$	$7.3 \le x_4 \le 8.3$
x_5	Normal	$\mu = 7.8,\ \sigma = 0.005$	$7.3 \le x_5 \le 8.3$
x_6	Normal	$\mu = 3.4,\ \sigma = 0.005$	$2.9 \le x_6 \le 3.9$
x_7	Normal	$\mu = 5.25,\ \sigma = 0.005$	$5.0 \le x_7 \le 5.5$

The DDO optimum for this problem is found as in Table 5.17, where four active constraints are found.

Table 5.17 Summary of DDO Result

Obj. Function Value	Design Point	Active Constraints
2994.4	(3.5000, 0.7000, 17.0000, 7.3934, 7.7153, 3.3506, 5.2866)	5, 6, 8, 11

Without the variable screening, the RBDO optimum point and the objective function value are found as in Table 5.18. There are four active constraints, and x_2, x_3 are at their design bounds. The computational cost of the RBDO for this problem is around 1 hour. The number of function evaluations is 371.

Table 5.18 Summary of RBDO Result Using All Variables

Variable Set	RBDO Optimum
x_1, x_2, x_3, x_4, x_5, x_6, x_7	$(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3651, 5.3017)$
Obj. Function Value	Computational Cost
3038.6	3609.35(s)
Active Constraints	Number of Function Calls
5, 6, 8, 11	371

To reduce the dimension of the problem, it is required to apply the screening method to the constraints 5, 6, 8, and 11. Using 100 samples to calculate the partial sample variance and $k = 50$ repetitions in the matrix in Eq. (3.5), Table 5.19 is obtained.

Table 5.19 Mean of Sample Variance in Iteration 0

Constraint 5						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Mean	0	7.75e-9	1.4e-11	7.8e-11	0	2.23e-5	0
Constraint 6						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Mean	0	1.1e-10	1.8e-13	0	9.3e-13	0	8.29e-6
Constraint 8						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Mean	2.14e-6	4.99e-5	0	0	0	0	0
Constraint 11						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Mean	0	0	0	0	4.25e-7	0	4.40e-7

The null and the alternative hypotheses for each constraint are set as

$$\text{Const } 5 - H_0 : \bar{s}_i \le (3.1895e-06) \times 100\%, \quad H_1 : \bar{s}_i > (3.1895e-06) \times 100\%$$
$$\text{Const } 6 - H_0 : \bar{s}_i \le (1.1851e-06) \times 100\%, \quad H_1 : \bar{s}_i > (1.1851e-06) \times 100\%$$
$$\text{Const } 8 - H_0 : \bar{s}_i \le (7.4361e-06) \times 100\%, \quad H_1 : \bar{s}_i > (7.4361e-06) \times 100\%$$
$$\text{Const } 11 - H_0 : \bar{s}_i \le (1.2375e-07) \times 100\%, \quad H_1 : \bar{s}_i > (1.2375e-07) \times 100\%$$

(5.5)

Eq. (5.5) is calculated from Eq. (3.10), by substituting k with 10, and $\bar{\mu}_i$ by the means in Table 5.19. The standard deviations of the partial sample variances are calculated in Table 5.20.

Table 5.20 Standard Deviation of Partial Sample Variance in Iteration 0

Constraint 5						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Standard Deviation	0	1.2e-09	2.0e-12	1.0e-11	0	2.4e-06	0
Constraint 6						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Standard Deviation	0	1.2e-11	1.2e-14	0	1.4e-13	0	5.6e-07
Constraint 8						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Standard Deviation	2.1e-07	1.0e-05	0	0	0	0	0
Constraint 11						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Standard Deviation	0	0	0	0	8.2e-08	0	5.9e-08

In this example, the sample sizes required to obtain a reasonable screening result was calculated from Eq. (4.4) as in Table 5.21 while setting $Z_\alpha = \Phi^{-1}(0.95)$,

$Z_\beta = \Phi^{-1}(0.5)$ where Φ is the CDF of the standard normal distribution. With the sample

size determined in Table 5.21, it is possible to calculate the test statistics.

Table 5.21 Sample Size Determination in Iteration 1

	Variable	x_2	x_6
Constraint 5	Sample Size	1	3
	Maximum Sample Size	3	
	Variable	x_2	x_7
Constraint 6	Sample Size	1	2
	Maximum Sample Size	2	
	Variable	x_1	x_2
Constraint 8	Sample Size	1	2
	Maximum Sample Size	2	
	Variable	x_5	x_7
Constraint 11	Sample Size	1	3
	Maximum Sample Size	3	

The maximum sample size in Table 5.21 is the minimum sample size to avoid the

errors with prescribed probability for each constraint. The final result of the variable

screening is shown in Table 5.22 – 5.25 below.

Table 5.22 Final Result of the Variable Screening for Constraint 5

Constraint 5							
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Test Statistic	-Inf	-3.9e+3	-3.7e+6	-9.3e+5	-Inf	11.3219	-Inf

Table 5.23 Final Result of the Variable Screening for Constraint 6

Constraint 6						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Test Statistic	-Inf	-8.5e+4	-4.6e+7	-Inf	-7.2e+6	-Inf	10.3590

Table 5.24 Final Result of the Variable Screening for Constraint 8

Constraint 8						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Test Statistic	-58.615	9.1456	-Inf	-Inf	-Inf	-Inf	-Inf

Table 5.25 Final Result of the Variable Screening for Constraint 11

Constraint 11						
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Test Statistic	-Inf	-Inf	-Inf	-Inf	26.016	-Inf	15.721

According to the result in Table 5.22 – 5.25, it is possible to consider x_6 from the constraint 5, x_7 from the constraint 6, x_2 from the constraint 8, and x_5, x_7 from the constraint 11 as the important variables. Because the design variable x_2 is on its lower bound, feasible domain might not exist for the RBDO. In fact, when the RBDO is carried out only with x_2, x_5, x_6, and x_7, which are selected design variables in Table 5.22 – 5.25, feasible domain does not exist so the RBDO could not find an optimum point. Consequently, the variable screening must be carried out for the rest of the variables in

the constraint 8. When the variable screening is complete, x_1 becomes a selected variable. Therefore, the selected design variable set is x_1, x_2, x_5, x_6 and x_7. Thus, the dimension of the problem shrinks to 5 from 7. Carrying out the RBDO with the selected design variables, Table 5.26 is obtained.

Table 5.26 Summary of RBDO Result Using Selected Design Variables

Variable Set	RBDO Optimum
x_1, x_2, x_5, x_6, x_7	$(3.5753, 0.7000, 17.0000, 7.3000, 7.7614, 3.3653, 5.3020)$
Obj. Function Value	Computational Cost
3039.4	2798.01(s)
Active Constraints	Number of Function Calls
5, 6, 8	218

Table 5.27 shows the calculated value of the probability of failure of each constraint function at the RBDO optimum point.

Table 5.27 Probability of Failure of Constraints at RBDO Optimum

Constraint	Probability of Failure (%)	Constraint	Probability of Failure (%)
G_1	0.000	G_7	0.000
G_2	0.000	G_8	0.154
G_3	0.000	G_9	0.000
G_4	0.000	G_{10}	0.000
G_5	0.140	G_{11}	0.000
G_6	0.107		

If x_3 and x_4 are considered to be deterministic, the probability of failure corresponding to the RBDO optimum in Table 5.26 for each constraint function satisfies the target probability of failure $P_f = 0.135\%$ in Eq. (5.4).

5.3. Cantilever Beam Problem Using ANSYS

The next example is an optimization problem of cantilever beam using implicit surrogate model. In this problem, the volume of the cantilever beam is the objective function to be minimized, while the displacement at the tip and the stress at the root are required to be smaller than a prescribed value. The problem is formulated as

$$\text{Minimize} \quad V = \left(x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + x_6\right)\left(cm^3\right)$$
$$\text{S.T.} \quad P\left[d_{491}(\mathbf{x}) > 0.0005cm\right] < 0.02275 \quad (5.6)$$
$$P\left[s_1(\mathbf{x}) > 300\,N/cm^2\right] < 0.02275$$

The beam is comprised of five sections; with the design variables are the heights of the sections. In Eq. (5.6), the subscripts in the constraints indicate the node number. The nodes are located on the beam model as shown in the Fig. 5.2. Young's modulus and Poisson's ratio used for this beam are listed in Table 5.28.

Table 5.28 Material Property

Young's Modulus	Poisson's Ratio
$21,000,000N/cm^2$	0.3

At the initial design stage, the dimension of the beam is $\left(L \times H\right) = \left(50cm \times 10cm\right)$. Further, the thickness of the beam is $1cm$ and the beam is not influenced by gravity. The

root of the beam on the left side is fixed along the edge, and force of $10N$ is applied along the right edge of the beam as shown in Fig. 5.2.
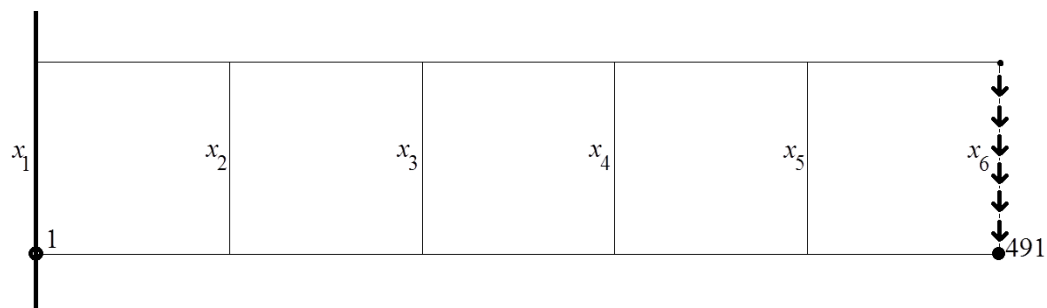
Figure 5.2 Design Variables and Location of Nodes on Cantilever Beam

The beam FEA model with 500 nodes is used for the stress and displacement analyses. At the initial design, the design variables are set as in Table 5.29. All the variables follow normal distribution with prescribed mean and standard deviation. The mean of the distribution is set as the design vector.

Table 5.29 Input Design Variable Information

Variable	Distribution Type	Initial Design Parameters	Design Bounds
x_1	Normal	$\mu = 10,\ \sigma = 0.2$	$5.0 \le x_1 \le 25.0$
x_2	Normal	$\mu = 10,\ \sigma = 0.2$	$1.0 \le x_2 \le 13.0$
x_3	Normal	$\mu = 10,\ \sigma = 0.2$	$1.0 \le x_3 \le 13.0$
x_4	Normal	$\mu = 10,\ \sigma = 0.2$	$1.0 \le x_4 \le 13.0$
x_5	Normal	$\mu = 10,\ \sigma = 0.2$	$1.0 \le x_5 \le 13.0$
x_6	Normal	$\mu = 10,\ \sigma = 0.2$	$1.0 \le x_6 \le 13.0$

The DDO optimum design is found as shown in Table 5.30, where both constraints are active. The design variable x_6 hit the lower bound since small x_6 does not reduce strength of the beam much whereas makes volume of the beam to be smaller efficiently.

Table 5.30 Summary of DDO Result

Objective Function Value	Design Point	Active Constraints
707.7776	$(10.9209, 9.5595, 8.3181, 6.7525, 4.7983, 1.0000)$	1, 2

The shape of the cantilever beam at the DDO optimum point is shown on Fig. 5.3. The tip of the beam becomes narrow, while the root of the beam becomes wide.
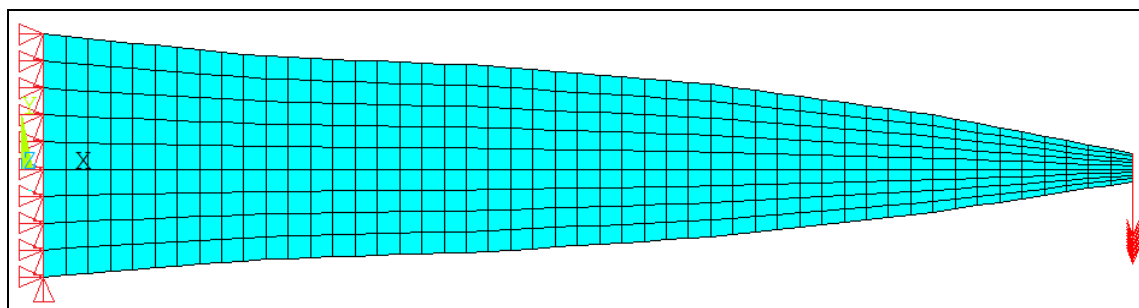
Figure 5.3 Shape of Beam at DDO Optimum

With all the six design variables, the RBDO optimum design is found as shown in Table 5.31. There is no significant change in the shape of the beam, but the height of each section becomes larger to satisfy the probabilistic constraints as shown in Fig. 5.4.
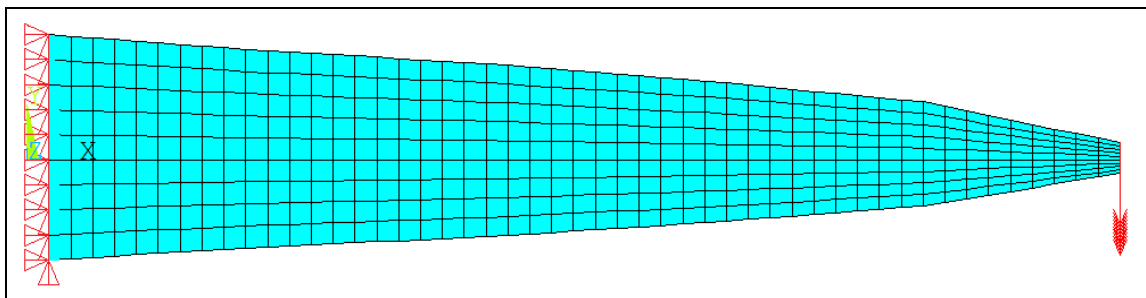
Figure 5.4 Shape of Beam at RBDO Optimum

Table 5.31 Summary of RBDO Result Using All Variables

Variable Set	RBDO Optimum
x_1, x_2, x_3, x_4, x_5, x_6	$(11.3335, 9.6370, 8.5635, 6.8662, 5.0435, 1.0000)$
Obj. Function Value	Computational Cost
725.5397	3674.280(s)
Active Constraints	Number of Function Calls
1, 2	176

Also, the probability of failure at the optimum is shown in Table 5.32. Both constraints satisfy the prescribed probability of failure.

Table 5.32 Probability of Failure at RBDO Optimum

	Constraint 1	Constraint 2
Probability of Failure	2.0323%	2.2789%

To reduce the dimension of the problem, the variable screening is carried out for

both active constraints.

Table 5.33 Final Result of Variable Screening

Constraint 1						
Variable	x_1	x_2	x_3	x_4	x_5	x_6
Test Statistic	-72.778	10.264	2.994	4.102	3.437	-97.885
Constraint 2						
Variable	x_1	x_2	x_3	x_4	x_5	x_6
Test Statistic	16.098	-185.250	-1.1E+05	-Inf	-Inf	-Inf

In constraint 1, the variables x_2, x_3, x_4 and x_5, which are in between the tip and the root, are selected as important variables. This is because output variance is affected by these design variables at the DDO optimum.

From the result of Table 5.33, the variables x_1, x_2, x_3, x_4 and x_5 are chosen as selected variables. Carrying out RBDO only with the selected variables, the result in Table 5.34 is obtained.

Table 5.34 RBDO Result Using Five Selected Variables

Selected Variable Set	RBDO Optimum
x_1, x_2, x_3, x_4, x_5	$(11.3366, 9.6632, 8.5037, 6.9120, 5.0065, 1.0000)$
Obj. Function Value	Computational Cost
725.0764	3201.413(s)
Active Constraints	Number of Function Calls
1, 2	168

As shown in Table 5.34, the computational cost decreases while the objective function value is same compared to Table 5.31. At this RBDO optimum, the probability of failure for each constraint is shown in Table 5.35. And the probability satisfies target value of 2.275% within acceptable range.

Table 5.35 Probability of Failure at RBDO Optimum Using Five Selected Variables

	Constraint 1	Constraint 2
Probability of Failure	2.3316%	2.2908%

More studies are carried out to find impact of different input standard deviation and the result of the variable screening. If the input standard deviation of a single variable (x_1) becomes larger as shown in Table 5.36, the ranking of the important variable is changed.

Table 5.36 Result of Variable Screening with Increased Standard Deviation of x_1

Variable	x_1	x_2	x_3	x_4	x_5	x_6
Standard Deviation	0.4	0.2	0.2	0.2	0.2	0.2
Test Statistic for Con. 1	5.997	8.2055	4.4207	5.898	8.241	-254.165
Test Statistic for Con. 2	24.112	-132.422	-1.2e+05	-2.3e+05	-7.4e+05	-6.2e+05
Result	Selected Variables		x_1, x_2, x_3 x_4, x_5	Screened Variables		x_6

Same problem is considered when the variance of x_2 is increased to 0.4. Then x_2 becomes a selected design variable for the constraint 1 as shown in Table 5.37.

Table 5.37 Result of Variable Screening with Increased Standard Deviation of x_2

Variable	x_1	x_2	x_3	x_4	x_5	x_6
Standard Deviation	0.2	0.4	0.2	0.2	0.2	0.2
Test Statistic for Con. 1	-60.426	5.648	-3.778	-3.731	-2.529	-985.662
Test Statistic for Con. 2	18.774	-155.12	-4.4e+05	-1.1e+05	-3.0e+05	-6.7e+05
Result	Selected Variables		x_1, x_2	Screened Variables		x_3, x_4, x_5, x_6

In case of the x_3 and x_4, if the input standard deviation is increased, x_1 and the variable with the increased input standard deviation are chosen as selected variables.

Table 5.38 Result of Variable Screening with Increased Standard Deviation of x_3

Variable	x_1	x_2	x_3	x_4	x_5	x_6
Standard Deviation	0.2	0.2	0.4	0.2	0.2	0.2
Test Statistic for Con. 1	-31.947	-5.962	10.017	-1.522	-0.882	-151.379
Test Statistic for Con. 2	29.516	-114.999	-1.1e+06	-2.4e+06	-1.3e+06	-0.1e+06
Result	Selected Variables		x_1, x_3	Screened Variables		x_2, x_4, x_5, x_6

Table 5.39 Result of Variable Screening with Increased Standard Deviation of x_4

Variable	x_1	x_2	x_3	x_4	x_5	x_6
Standard Deviation	0.2	0.2	0.2	0.4	0.2	0.2
Test Statistic for Con. 1	-65.429	-12.909	-4.527	6.002	-6.815	-96.872
Test Statistic for Con. 2	14.917	-152.789	-7.0e+05	-2.4e+05	-1.4e+06	-2.4e+05
Result	Selected Variables		x_1, x_4	Screened Variables		x_2, x_3 x_5, x_6

If the input standard deviation of a single variable is large enough as shown in Table 5.37 – 5.39, only one variable is selected. In this case, the variable dominates the output variance, which means considering the variable in the design stage will be the fastest way to satisfy the probabilistic constraints. If the input standard deviation is not large enough that a single variable does not dominate the output variance; and not small enough that nearly all the variables are selected, then multiple number of design variables are selected as shown in Table 5.40.

Table 5.40 Result of Variable Screening with Increased Standard Deviation of x_5

Variable	x_1	x_2	x_3	x_4	x_5	x_6
Standard Deviation	0.2	0.2	0.2	0.2	0.33	0.2
Test Statistic for Con. 1	-427.806	3.885	-8.524	-3.033	84.592	-787.372
Test Statistic for Con. 2	6.747	-95.097	-2.9e+05	-12e+05	-4.3e+05	-4.0e+05
Result	Selected Variables		x_1, x_2, x_5	Screened Variables		x_3, x_4, x_6

In Table 5.40, the variable x_5 with increased standard deviation becomes a selected variable, but another variable x_2 is also selected at the same time.

Though standard deviation of x_6 is increased, however, the variable is always screened out. This is because neither of the constraints are a function of x_6. And this is also the reason why x_6 hits the lower bound. Table 5.41 shows the variable screening result with increased input standard deviation of x_6.

Table 5.41 Result of Variable Screening with Increased Standard Deviation of x_6

Variable	x_1	x_2	x_3	x_4	x_5	x_6
Standard Deviation	0.2	0.2	0.2	0.2	0.2	0.4
Test Statistic for Con. 1	-170.047	13.509	9.023	9.132	12.649	-19.778
Test Statistic for Con. 2	28.433	-155.52	-4.9e+05	-9.5e+05	-15e+05	-0.3e+05
Result	Selected Variables		x_1, x_2, x_3 x_4, x_5	Screened Variables		x_6

Likewise, all the variables other than x_1 are screened out for the constraint 2. The reason is because the second constraint is only a function of x_1. Therefore, even though a function is the one of unknown number of variables, the screening method automatically filters the variables which are not employed in the function. Once the variables are selected, the computational cost will significantly decrease when carrying out the RBDO. Table 5.42 shows the RBDO result the corresponding computational time using the

selected design variables obtained in Table 5.37 – 5.40 During the RBDO procedure, the screened variables remain at the DDO optimum value.

Table 5.42 RBDO Result Using Selected Variables

Selected Variable Set	RBDO Optimum	Obj. Function Value
No Screening	$(11.0255, 9.6035, 8.3531, 6.7648, 4.8682, 1.0000)$	712.0487
x_1, x_2	$(11.3542, 9.8017, 8.3181, 6.7525, 4.7983, 1.0000)$	716.9561
x_1, x_3	$(11.3179, 9.5595, 8.5832, 6.7525, 4.7983, 1.0000)$	716.9535
x_1, x_4	$(11.3177, 9.5595, 8.3181, 7.0125, 4.7983, 1.0000)$	716.9453
x_1, x_2, x_5	$(11.3444, 9.7254, 8.3181, 6.7525, 5.0615, 1.0000)$	720.2817

As shown in Table 5.42, the objective function values at each RBDO optimum when the screening method is used are slightly larger than that of the RBDO optimum without the variable screening. However, when the computational costs are compared, a significant reduction is observed.

Table 5.43 Computational Cost Using Selected Variables

Selected Variable Set	Computational Cost (s)
No Screening	3674.280
x_1, x_2	581.062
x_1, x_3	579.747
x_1, x_4	625.194
x_1, x_2, x_5	1285.527

Though the variable screening method considers the screened variables as deterministic, the screened variables are also random in reality. Therefore, when the reliability analysis for each RBDO optimum is carried out when setting all the design variables are random, it is possible to notice that the target probability of failure of constraints in Eq. (5.6) are not satisfied as shown in Table 5.44.

Table 5.44 Probability of failure of Each Constraint with Selected Design Variables

Selected Variable Set	Constraint 1	Constraint 2
No Screening	2.279%	2.240%
x_1, x_2	18.154%	2.301%
x_1, x_3	24.911%	2.344%
x_1, x_4	26.484%	2.407%
x_1, x_2, x_5	11.6441%	2.315%

Because the number of variables one can deal with is limited, it is not possible to satisfy the probabilistic constraints in Eq. (5.6). However, the RBDO result using the selected variable set gives the lowest probability of failure at the RBDO optimum when compared to other combinations of the same number of the design variables. For example, when the input standard deviation of x_2 is increased, the selected design variables are x_1 and x_2. When different variable sets are used to obtain the DDO optimum, the probability of failure at the RBDO optimum is worse than that of the RBDO optimum using x_1 and x_2. Table 5.45 shows the RBDO result with different combinations and Table 5.46 shows the result of the reliability analysis at each RBDO optimum.

Table 5.45 RBDO Optimum using Different Design Variable Combinations

Combinations	RBDO Optimum
Selected Variables (x_1, x_2)	$(11.3542, 9.8017, 8.3181, 6.7525, 4.7983, 1.0000)$
x_1, x_3	$(11.3193, 9.5595, 8.5796, 6.7525, 4.7982, 1.0000)$
x_1, x_4	$(11.3178, 9.5595, 8.3181, 7.0124, 4.7983, 1.0000)$
x_1, x_5	$(11.3196, 9.5595, 8.3181, 6.7525, 5.0623, 1.0000)$
x_1, x_6	$(11.3635, 9.5595, 8.3181, 6.7525, 4.7983, 1.1194)$

Table 5.46 Probability of Failure of Constraints with Different Design Variable Combinations

Combinations	Constraint 1	Constraint 2
Selected Variables (x_1, x_2)	18.154%	2.301%
x_1, x_3	23.681%	2.694%
x_1, x_4	23.961%	2.734%
x_1, x_5	23.490%	2.678%
x_1, x_6	35.302%	1.594%

Therefore, the screening method finds most efficient set of variables for more reliable design when the limited computation power is available. The rest of the combinations could not find the RBDO optimum because the second constraint function is a function of x_1 only. It is noticeable in Table 5.42 that when x_1 and x_6 are used to carry out the RBDO, the probability of failure for the constraint 2 even becomes smaller. This is because the constraint 1 is a function of x_1 to x_5, and when (x_1, x_2), (x_1, x_3),

(x_1, x_4), or (x_1, x_5) are used, there are two design variables to adjust the constraint 1.

But when (x_1, x_6) are used, there is one design variable to adjust the constraint 1.

Therefore, x_1 becomes larger than that of other combinations to compensate the effect

from either x_2, x_3, x_4 or x_5.

CHAPTER VI

CONCLUSION

In general, the screening method is applied to a design optimization problem to rank the order of the design variables so that the problem size could be reduced. So far, it is shown that the screening method shown in this research considering the output variance of a function works well as demonstrated with the suitable examples. One of the benefits of the method is that it could be widely applicable because the method only requires the MCS of the output value of a function, which makes it possible to use for responses with implicit functions. The method is new in a sense because it considers not only the shape of a function, but also the input standard deviation of the design variables.

The test-of-hypothesis is utilized for the screening method. Because the test requires a prescribed threshold value for the critical region, the measurement for ranking variables, such as the threshold value for the region boundary, could be defined differently according to the problem involved. And this fact makes the screening method more flexible.

Another strong point of the proposed method is found in the number of function evaluations. Even though the computation cost is affected by the number of MCS points, repetitions, iterations, design variables, and active constraints as in Eq. (3.14), the number does not necessarily exceed the number of MCS in the design stage because only the DDO optimum point is involved in the screening method. Moreover, the number of function evaluation is determined in a way that the computational cost is minimized while maintaining a reasonable confidence level as shown in the chapter 4, and thus the computational cost for this method is relatively small.

On the other hand, the result of the proposed screening method is valid only around the DDO optimum design where the test is carried out. Because the function value changes as design moves, if a design variable moves significantly then the ranking of the design variable could be changed. Therefore, if a constraint is highly nonlinear around the DDO optimum design, then there is a possibility that the ranking determined at the DDO optimum design will be changed while the design moves toward the RBDO optimum design.

Another problem comes from the relative effect of the variables. Because the method compares not the absolute value but the relative value of the output variance, the screening method will not work appropriately if one variable has significantly large impact on the output variance. In this case, only the variable which gives large output variance will be reported as an important one while the rest of the variables are all screened out, which could be an undesirable result.

A problem regarding active constraints is also a concern. In a special case, it is possible that the limit states of the inactive constraints at DDO optimum during the RBDO design process are so close to the DDO optimum design that the inactive constraints become active. In this case, the screening method should be applied to the inactive ones as well as active constraints. However, it is not easy to anticipate which constraint will become active in advance.

REFERENCES

[1] Arora, J. S., *Introduction to Optimum Design*, 2nd ed., Elsevier Inc., San Diego, CA, 2004.

[2] Tu, J., Choi, K. K., and Park, Y. H., "A New Study on Reliability-Based Design Optimization," *Journal of Mechanical Design*, Vol. 121, pp. 557-564, 1999.

[3] Lee, I., Choi, K. K., and Gorsich, D., "Equivalent Standard Deviation to Convert High-reliability Model to Low-reliability Model for Efficiency of Sampling-based RBDO," *37th ASME Design Automation Conference*, Washington, D.C., August 28-31, 2011.

[4] Chen, W., Jin, R., and Sudjianto, A., "Analytical Variance-Based Global Sensitivity Analysis in Simulation-Based Design Under Uncertainty," *Journal of Mechanical Design*, Vol. 127, pp. 875-886, 2005.

[5] Craig, K. J., Stander, N., Dooge, D. A., and Varadappa, S., "Automotive Crashworthiness design using response surface-based variable screening and optimization," *Engineering Computations: International Journal for Computer-Aided Engineering and Software*, Vol. 22, No.1, pp. 38-61, 2005.

[6] Wu, Y-T. and Mohanty, S., "Variable screening and ranking using sampling-based sensitivity measures," *Reliability Engineering and System Safety*, 91 pp. 634-647, 2006.

[7] Zhao, L., Choi, K. K., and Lee, I., "A Metamodeling Method Using Dynamic Kriging for Design Optimization," *AIAA Journal*, Vol. 49, No. 9, pp. 2034-2046, 2011.

[8] Rubinstein, R. Y., and Shapiro, A., *Discrete Event Systems – Sensitivity Analysis and Stochastic Optimization by the Score Function Method*, Wiley, New York, 1993.

[9] Rahman, S., "Stochastic Sensitivity Analysis by Dimensional Decomposition and Score Functions," *Probab. Eng. Mech.*, Vol. 24, pp. 278-287, 2009.

[10] Lee, I., Choi, K. K., Noh, Y. J., Zhao, L., and Gorsich, D., "Sampling-Based Stochastic Sensitivity Analysis Using Score Functions for RBDO Problems With Correlated Random Variables," *Journal of Mechanical Engineering*, Vol. 133, pp.021003-01-021003-10, 2011.

[11] Huntington, D. E. and Lyrintzis, C. S., "Improvements to and limitations of Latin hypercube sampling," *Prob. Engng. Mech*, Vol. 13, No. 4, pp. 245-253, 1998.

[12] Owen, A. B., "A Central Limit Theorem for Latin Hypercube Sampling," *J.R. Statist. Soc. B*, Vol. 54, No. 2, pp. 541-551, 1992.

[13] Haldar, A., and Mahadevan, S., *Probability, reliability, and statistical methods in engineering design*, John Wiley, New York, 2000.

[14] Rust, J., "Using Randomization to Break the Curse of Dimensionality," *Econometrica*, Vol. 65, No. 3, pp. 487-516, 1997.

[15] Chen, W., Jin, R. and Sudjianto, A., "Analytical Metamodel-Based Global Sensitivity Analysis and Uncertainty Propagation for Robust Design," *SAE international*, 2004-01-0429, 2004.

[16] Robert V. Hogg, Joseph W. McKean, and Allen T. Craig, *Introduction to Mathematical Statistics*, Pearson Education Inc., New Jersey, 2005.

[17] Lee, J. J. and Lee, B. C., "Efficient Evaluation of Probabilistic Constraints Using an Envelope Function," *Engineering Optimization*, Vol. 37, No. 2, pp. 185-200, 2005.